Recent progress in the theory of homogenization with oscillating Dirichlet data

David Gérard-Varet*and Nader Masmoudi†

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In this talk we study the homogenization of elliptic systems with Dirichlet boundary condition, when both the coefficients and the boundary datum are oscillating, namely ε-periodic. In particular, in the paper [9], we showed that, as $\varepsilon \to 0$, the solutions converge in L^2 with a power rate in ε, and we identified the homogenized limit system and the homogenized boundary data. Due to a boundary layer phenomenon, this homogenized system depends in a non trivial way on the boundary. The analysis in [9] answers a longstanding open problem, raised for instance in [4].

1 Introduction

Homogenization of elliptic systems arises in several physical problems where a mixture is present. Some of the main applications of the theory are the diffusion of heat or electricity in a non-homogeneous media, the theory of elasticity of mixtures, ... Physically, the main goal of the theory is to try to compute accurate and effective properties of these mixtures. Mathematically, we have to find a limit system towards which the solutions of homogenization problem converge. This passage from "microscopic" to "macroscopic" description is called in the literature "homogenization".

When both the coefficients of the system and the boundary datum are oscillating (ε-periodic) and due to a boundary layer phenomenon, this homogenized system depends in a non trivial way on the boundary. In this talk, we answer a longstanding open problem, raised for instance by Bensoussan, Lions and Papanicolaou in their book "Asymptotic analysis for periodic structures" [4, page xiii]:

> *Of particular importance is the analysis of the behavior of solutions near boundaries and, possibly, any associated boundary layers. Relatively little seems to be known about this problem.*

In particular this result extends substantially previous works obtained for polygonal domains with sides of rational slopes as well as our previous paper [8] where the case of irrational slopes was considered. We hope that these notes give a better understanding of the proof of the result in [9].

*Institut de Mathématiques de Jussieu and University Paris 7, 175 rue du Chevaleret,75013 Paris, FRANCE.
D. G-V is partially supported by the project Instabilities in Hydrodynamics
†Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA.
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2 The homogenization problem

We consider the homogenization of elliptic systems in divergence form

$$-\nabla\cdot\left(A\left(\cdot/\varepsilon\right)\nabla u\right)(x)=f,\quad x\in\Omega, \tag{2.1}$$

set in a bounded domain Ω of $\mathbb{R}^d$, $d\geq 2$, with an oscillating Dirichlet data

$$u(x)=\varphi(x,x/\varepsilon),\quad x\in\partial\Omega. \tag{2.2}$$

As is customary, $\varepsilon>0$ is a small parameter, and $A=A(y)$ takes values in $M_d\left(M_N(\mathbb{R})\right)$, namely $A^{\alpha\beta}(y)\in M_N(\mathbb{R})$ is a family of functions of $y\in\mathbb{R}^d$, indexed by $1\leq\alpha,\beta\leq d$, with values in the set of $N\times N$ matrices. Here, $u=u(x)$ and $\varphi=\varphi(x,y)$ take their values in $\mathbb{R}^N$. We recall, using Einstein convention for summation, that for each $1\leq i\leq N$,

$$(\nabla\cdot A\left(\cdot/\varepsilon\right)\nabla u)_i(x)\ :=\ \partial_{x_\alpha}\left[A^{\alpha\beta}_{ij}\left(\cdot/\varepsilon\right)\,\partial_{x_\beta}u_j\right](x).$$

In the sequel, Greek letters $\alpha,\beta,...$ will range between 1 and d and Latin letters $i,j,k,...$ will range between 1 and N.

In the context of thermics, $d=2$ or 3, $N=1$, u is the temperature, and $\sigma=A(\cdot/\varepsilon)\nabla u$ is the heat flux given by Fourier law. The parameter ε models heterogeneity, that is short-length variations of the material conducting properties. The boundary term φ in (2.2) corresponds to a prescribed temperature at the surface of the body and f is a source term. In the context of linear elasticity, $d=2$ or 3, $N=d$, u is the unknown displacement, f is the external load and A is a fourth order tensor that models Hooke's law.

We make three hypotheses:

i) Ellipticity: For some $\lambda>0$, for all family of vectors $\xi=\xi^\alpha_i\in\mathbb{R}^{Nd}$

$$\lambda\sum_\alpha\xi^\alpha\cdot\xi^\alpha\ \leq\ \sum_{\alpha,\beta,i,j}A^{\alpha,\beta}_{ij}\,\xi^\beta_j\,\xi^\alpha_i\ \leq\ \lambda^{-1}\sum_\alpha\xi^\alpha\cdot\xi^\alpha.$$

ii) Periodicity: $\forall y\in\mathbb{R}^d$, $\forall h\in\mathbb{Z}^d$, $\forall x\in\partial\Omega$, $A(y+h)\ =\ A(y)$, $\quad\varphi(x,y)\ =\ \varphi(x,y+h)$.

iii) Smoothness: The functions A,f and φ, as well as the domain Ω are smooth. It is actually enough to assume that φ and Ω are in some H^s for s big enough, but we will not try to compute the optimal regularity.

The main question we are trying to answer is the following:

Question: What is the limit behavior of the solutions u^ε as $\varepsilon\to 0$? Can we go beyond the limit and compute a full expansion of u^ε ?

This question goes back at least to the 1970's, and a classical approach consists in trying a two-scale expansion:

Classical approach: *Two-scale asymptotic expansion*:

$$\boxed{u^\varepsilon_{app}\ =\ u^0(x)\ +\ \varepsilon u^1(x,x/\varepsilon)\ +\ \ldots\ +\ \varepsilon^n u^n(x,x/\varepsilon)} \tag{2.3}$$

with $u^i=u^i(x,y)$ periodic in y.

3 Case without boundary

The two-scale approach works well in the case without boundary, namely in the whole space case or in the case of a periodic domain (say of period 1 and ε is taken to be equal to $1/n$ with n an integer). In particular one can construct inductively all the terms in the expansions. Let us recall few classical facts (see for instance [23, 20, 13, 6]) :

i) The construction of the u^i's involves the famous *cell problem*

$$\boxed{-\nabla \cdot \, (A\nabla\chi^\gamma)\,(y) \;=\; \nabla_\alpha \cdot A^{\alpha\gamma}\,(y), \quad y \text{ in } \mathbb{T}^d} \tag{3.1}$$

with solution $\chi^\gamma \in M_N(\mathbb{R})$.

ii) The solvability condition for u^2 yields the equation satisfied by u^0, namely u^0 (which does not depend on y) satisfies

$$\nabla \cdot A^0\nabla u^0 \;=\; f \tag{3.2}$$

where the constant homogenized matrix is given by

$$\boxed{A^{0,\alpha\beta} \;=\; \int_{\mathbb{T}^d} A^{\alpha\beta}(y)\,dy \;+\; \int_{\mathbb{T}^d} A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y)\,dy.}$$

The second term in the expansion (2.3) reads

$$u^1(x,y) \;:=\; \tilde{u}^1(x,y) + \bar{u}^1(x) \;:=\; -\chi^\alpha(y)\partial_{x_\alpha}u^0(x) + \bar{u}^1(x), \tag{3.3}$$

where χ is again the solution of (3.1).

To find an equation for the average part $\bar{u}^1(x)$, one needs to introduce another family of 1-periodic matrices

$$\Upsilon^{\alpha\beta} = \Upsilon^{\alpha\beta}(y) \in M_n(\mathbb{R}),\ \alpha,\beta = 1,\dots,d,$$

satisfying

$$-\nabla_y \cdot A\,\nabla_y\Upsilon^{\alpha\beta} \;=\; B^{\alpha\beta} - \int_y B^{\alpha\beta}, \quad \int_y \Upsilon^{\alpha\beta} = 0, \tag{3.4}$$

where

$$B^{\alpha\beta} \;:=\; A^{\alpha\beta} - A^{\alpha\gamma}\frac{\partial\chi^\beta}{\partial y_\gamma} - \frac{\partial}{\partial y_\gamma}\left(A^{\gamma\alpha}\chi^\beta\right).$$

Formal considerations yield

$$u^2(x,y) \;:=\; \Upsilon^{\alpha,\beta}\frac{\partial^2 u^0}{\partial x_\alpha \partial x_\beta} \;-\; \chi^\alpha\partial_\alpha\bar{u}^1 + \bar{u}^2 \tag{3.5}$$

and that the average term $\bar{u}^1 = \bar{u}^1(x)$ formally satisfies the equation

$$-\nabla \cdot A^0\nabla\bar{u}^1 = c^{\alpha\beta\gamma}\frac{\partial^3 u^0}{\partial x_\alpha\partial x_\beta\partial x_\gamma}, \quad c^{\alpha\beta\gamma} \;:=\; \int_y A^{\gamma\eta}\frac{\partial\Upsilon^{\alpha\beta}}{\partial y_\eta} - A^{\alpha\beta}\chi^\gamma. \tag{3.6}$$

We refer to [2] for more details.

Inductively, one can keep constructing all the terms of the expansion by introducing new corrector families as in (3.4) and solving homogenized systems to determine $\bar{u}^k$ as in (3.6). Note that in this case, we do not need an extra boundary condition to solve (3.6).

4 Case with boundary

Two boundary conditions have been widely studied and are by now well understood as long as we are only interested in the first term of the expansion:

1. *The non-oscillating Dirichlet problem*, that is (2.1) and (2.2) with $\varphi = \varphi(x)$.
2. *The oscillating Neumann problem*, that is (2.1) and

$$n(x) \cdot (A(\cdot/\varepsilon)\nabla u)\,(x) \;=\; \varphi(x, x/\varepsilon), \quad x \in \partial\Omega, \tag{4.1}$$

   where $n(x)$ is the normal vector and with a standard compatibility condition on φ. Note that in thermics, this boundary condition corresponds to a given heat flux at the solid surface.

Notice that in both problems, the usual energy estimate provides a uniform bound on the solution u^ε in $H^1(\Omega)$.

For the non-oscillating Dirichlet problem, one shows that u^ε weakly converges in $H^1(\Omega)$ to the solution u^0 of the homogenized system

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = f, & x \in \Omega, \\ u^0(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{4.2}$$

It is also proved in [4] that

$$u^\varepsilon(x) \;=\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; O(\sqrt{\varepsilon}), \;\text{ in }\; H^1(\Omega). \tag{4.3}$$

Actually, an open problem in this area was to compute the next term in the expansion in the presence of a boundary, namely to compute $u^1(x, x/\varepsilon)$. Indeed, it is not difficult to see that

$$u^1(x, y) \;=\; -\chi^\alpha(y)\partial_{x_\alpha} u^0(x) + \bar{u}^1(x), \tag{4.4}$$

where $\bar{u}^1(x)$ solves the homogenized equation (3.6). However, the main difficulty is to find the boundary data for $\bar{u}^1(x)$. The new analysis of [9] gives an answer to this problem (see also next section).

For the oscillating Neumann problem, two cases must be distinguished. On one hand, if $\partial\Omega$ does not contain flat pieces, or if it contains finitely many flat pieces whose normal vectors do not belong to $\mathbb{R}\mathbb{Z}^n$, then

$$\varphi(\cdot, \cdot/\varepsilon) \to \overline{\varphi} := \int_{[0,1]^d} \varphi \;\text{ weakly in } L^2(\partial\Omega)$$

and u^ε converges weakly to the solution u^0 of

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ n(x) \cdot \left(A^0 \nabla u^0\right)(x) = \overline{\varphi}(x), & x \in \partial\Omega. \end{cases} \tag{4.5}$$

On the other hand, if $\partial\Omega$ does contain a flat piece whose normal vector belongs to $\mathbb{R}\,\mathbb{Q}^d$, then the family $\varphi(\cdot, \cdot/\varepsilon)$ may have a continuum of accumulation points as $\varepsilon \to 0$. Hence, u^ε may have a continuum of accumulation points in H^1 weak, corresponding to different Neumann boundary data. We refer to [4] for all details.

5 Case of an oscillating Dirichlet data

Here we study (2.1) with the boundary data (2.2). One of the motivation to study this case is actually to understand the boundary condition for $\bar{u}^1(x)$ which appears in (3.3).

Let us explain the two main sources of difficulties in studying (2.1)-(2.2):

i) One has uniform L^p bounds on the solutions u^ε of (2.1)-(2.2), but no uniform H^1 bound *a priori*. This is due to the fact that

$$\|x \mapsto \varphi(x, x/\varepsilon)\|_{H^{1/2}(\partial\Omega)} = O(\varepsilon^{-1/2}), \text{ resp. } \|x \mapsto \varphi(x, x/\varepsilon)\|_{L^p(\partial\Omega)} = O(1),\ p > 1.$$

The usual energy inequality, resp. the estimates in article [3, page 8, Thm 3] yields

$$\|u^\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^{-1/2}), \text{ resp. } \|u^\varepsilon\|_{L^p(\Omega)} = O(1),\ p > 1.$$

This indicates that singularities of u^ε are *a priori* stronger than in the usual situations. It is rigorously established in the core of the paper [9].

ii) Furthermore, one can not expect these stronger singularities to be periodic oscillations. Indeed, the oscillations of φ are at the boundary, along which they do not have any periodicity property. Hence, it is reasonable that u^ε should exhibit concentration near $\partial\Omega$, with no periodic character, as $\varepsilon \to 0$. This is a so-called *boundary layer phenomenon*. The key point is to describe this boundary layer, and its effect on the possible weak limits of u^ε.

It is important to note that there is also a boundary layer in the non-oscillating Dirichlet problem, although it has in this case a lower amplitude (it is only necessary to compute the boundary data of $\bar{u}^1$ to solve (3.6)). More precisely, it is responsible for the $O(\sqrt{\varepsilon})$ loss in the error estimate (4.3). If either the L^2 norm, or the H^1 norm in a relatively compact subset $\omega \Subset \Omega$ is considered, one may avoid this loss as strong gradients near the boundary are filtered out. Following Allaire and Amar (see [2, Theorem 2.3]), we can give a more precise description than (4.3):

$$u^\varepsilon \;=\; u^0(x) \;+\; O(\varepsilon) \;\text{ in }\; L^2(\Omega), \quad u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) \;+\; O(\varepsilon) \;\text{ in }\; H^1(\omega). \tag{5.1}$$

Still following [2], another way to put the emphasis on the boundary layer is to introduce the solution $u_{bl}^{1,\varepsilon}(x)$ of

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right) \nabla u_{bl}^{1,\varepsilon} = 0, & x \in \Omega \subset \mathbb{R}^d, \\ u_{bl}^{1,\varepsilon} = -u^1(x, x/\varepsilon), & x \in \partial\Omega. \end{cases} \tag{5.2}$$

Actually, understanding this system and requiring that $u_{bl}^{1,\varepsilon}$ goes to zero inside the domain Ω allows to determine the right boundary condition for $\bar{u}^1$. Hence, one can show that

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; O(\varepsilon), \;\text{ in }\; H^1(\Omega). \tag{5.3}$$

or

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; O(\varepsilon^2), \;\text{ in }\; L^2(\Omega). \tag{5.4}$$

Note that system (5.2) is a special case of (2.1)-(2.2). Thus, the homogenization of the oscillating Dirichlet problem may give a refined description of the non-oscillating one.

6 Prior results

Until recently, *results were all limited to convex polygons with rational normals.* This means that

$$\Omega \;:=\; \cap_{k=1}^{K} \left\{ x, \quad n^k \cdot x > c^k \right\}$$

is bounded by K hyperplanes, *whose unit normal vectors n^k belong to $\mathbb{R}\,\mathbb{Q}^d$.* Under this assumption, the study of (2.1)-(2.2) can be carried out. The keypoint is the addition of boundary layer correctors to the formal two-scale expansion:

$$u^\varepsilon(x) \;\sim\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; \sum_k v_{bl}^k \left(x, \frac{x}{\varepsilon}\right), \tag{6.1}$$

where $v_{bl}^k = v_{bl}^k(x,y) \in \mathbb{R}^n$ is defined for $x \in \Omega$, and y in the half-space

$$\Omega^{\varepsilon,k} \;=\; \left\{ y, \quad n^k \cdot y > c^k/\varepsilon \right\}.$$

These correctors satisfy

$$\left\{ \begin{aligned} -\nabla_y \cdot\; A(y) \nabla_y\, v_{bl}^k = 0, \quad y \in \Omega^{\varepsilon,k}, \\ v_{bl}^k = \varphi(x,y) - u^0(x), \quad y \in \partial\Omega^{\varepsilon,k}. \end{aligned} \right. \tag{6.2}$$

We refer to the papers by Moskow and Vogelius [19], and Allaire and Amar [2] for more details. These papers deal with the special case (5.2), but the results adapt to more general oscillating data. Note that x is just a parameter in (6.2) and that the assumption $n^k \in \mathbb{R}\,\mathbb{Z}^d$ yields periodicity of the function $A(y)$ tangentially to the hyperplanes. The periodicity property is used in a crucial way in the aforementioned references. First, it yields easily well-posedness of the boundary layer systems (6.2). Second, as was shown by Tartar in [18, Lemma 10.1] (see also subsection 7.2), the solution $v_{bl}^k(x,y)$ converges exponentially fast to some $v_{bl,*}^k(x) = \varphi_*^k(x) - u^0(x)$, when y goes to infinity transversely to the k-th hyperplane. In order for the boundary layer correctors to vanish at infinity (and to be $o(1)$ in L^2), one must have $v_{bl,*}^k = 0$, which provides the boundary condition for u^0. Hence, u^0 should satisfy a system of the type

$$\left\{ \begin{aligned} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = f, \quad x \in \Omega, \\ u^0(x) = \varphi_*(x), \quad x \in \partial\Omega. \end{aligned} \right. \tag{6.3}$$

where $\varphi_*(x) \;:=\; \varphi_*^k(x)$ on the k-th side of Ω. Nevertheless, this picture is not completely correct. Indeed, there is still *a priori* a dependence of φ_*^k on ε, through the domain $\Omega^{\varepsilon,k}$. In fact, Moskow and Vogelius exhibit examples for which there is an infinity of accumulation points for the φ_*^k's, as $\varepsilon \to 0$. Eventually, they show that the accumulation points of u^ε in L^2 are the solutions u^0 of systems like (6.3), in which the φ_*^k's are replaced by their accumulation points. See [19] for rigorous statements and proofs. We stress that their analysis relies heavily on the special shape of Ω, especially the rationality assumption.

A step towards more generality has been made in our recent paper [8] (see also [7]), in which generic convex polygonal domains are considered. Indeed, we assume in [8] that *the normals $n = n^k$ satisfy the Diophantine condition:*

$$\text{For all } \;\xi \in \mathbb{Z}^d \setminus \{0\} \quad |P_{n^\perp}(\xi)| > \kappa\, |\xi|^{-l}, \quad \text{for some } \;\kappa,\, l > 0, \tag{6.4}$$

where $P_{n^\perp}$ is the projector orthogonal to n. Note that for dimension $d = 2$ this condition amounts to:

$$\text{For all } \xi \in \mathbb{Z}^d \setminus \{0\} \quad |n^\perp \cdot \xi| \;:=\; |-n_2\xi_1 + n_1\xi_2| > \kappa\, |\xi|^{-l}, \quad \text{for some } \kappa,\, l > 0,$$

whereas for $d = 3$, it is equivalent to:

$$\text{For all } \xi \in \mathbb{Z}^d \setminus \{0\} \quad |n \times \xi| > \kappa\, |\xi|^{-l}, \quad \text{for some } \kappa,\, l > 0.$$

Condition (6.4) is generic in the sense that it holds for almost every $n \in S^{d-1}$.

Under this Diophantine assumption, one can perform the homogenization of problem (2.1)-(2.2). *Stricto sensu*, only the case (5.2), $d = 2, 3$ is treated in [8], but our analysis extends straightforwardly to the general setting. Despite a loss of periodicity in the tangential variable, we manage to solve the boundary layer equations, and prove convergence of v^k_{bl} away from the boundary. The main idea is to work with quasi-periodic functions instead of periodic ones (see also subsection 7.3). Interestingly, and contrary to the "rational case", the field φ^k_* does not depend on ε. As a result, we establish convergence of the whole sequence u^ε to the single solution u^0 of (6.3). We stress that, even in this polygonal setting, the boundary datum φ_* depends in a non trivial way on the boundary. In particular, it is not simply the average of φ with respect to y, contrary to what happens in the Neumann case.

7 Main new result and sketch of proof

The main new result of [9] is to treat the case of a smooth domain:

Theorem 1 (Homogenization in smooth domains)
Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \geq 2$. We assume that it is uniformly convex (all the principal curvatures are bounded from below).

Let u^ε be the solution of system (2.1)-(2.2)*, under the ellipticity, periodicity and smoothness conditions i)-iii).*

There exists a boundary term φ_* (depending on φ, A and Ω), with $\varphi_* \in L^p(\partial\Omega)$ for all finite p, and a solution u^0 of (6.3)*, with $u^0 \in L^p(\Omega)$ for all finite p, such that:*

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \;\leq\; C_\alpha\, \varepsilon^\alpha, \qquad \textit{for all } 0 < \alpha < \frac{d-1}{3d+5}. \tag{7.1}$$

We will present a sketch of the proof of theorem 1:

¿From the two difficulties explain in section 5, we know that the first term in the expansion (2.3) should be independent of y and should solve (3.2). The main question is :

Question: What is the boundary value φ^0 of u^0 ?

Solution: We need a boundary layer corrector

Difficulty: There is no clear structure for the boundary layer.

Guess: The boundary layer has typical scale ε and there are no curvature effect:

- Near a point $x_0 \in \partial\Omega$, we replace $\partial\Omega$ by the tangent plane at x_0:

$$T_0(\partial\Omega) \;:=\; \{x,\; x \cdot n_0 = x_0 \cdot n_0\}$$

- We dilate by a factor ε^{-1}.

Formally, for $x \approx x_0$, one looks for

$$\boxed{u^{\varepsilon,bl}(x) \;\approx\; U_0(x/\varepsilon)}$$

where the profile $U_0 = U_0(y)$ is defined in the half plane

$$H_0^\varepsilon \;=\; \{y,\; y \cdot n_0 > \varepsilon^{-1} x_0 \cdot n_0\}.$$

It satisfies the system:

$$\boxed{\left\{\begin{aligned} \nabla_y \cdot (A \nabla_y U_0) = 0 \quad \text{in } H_0^\varepsilon, \\ U_0|_{\partial H_0^\varepsilon} = \varphi - \varphi^0(x_0). \end{aligned}\right.} \tag{7.2}$$

Notice that in this system, x_0 is just a parameter.

7.1 Study of an auxiliary boundary layer system

The previous heuristic justifies the study of

$$\boxed{\left\{\begin{aligned} \nabla_y \cdot (A \nabla_y U) = 0 \quad \text{in } H, \\ U|_{\partial H} = \phi. \end{aligned}\right.} \tag{BL}$$

where $H \;:=\; \{y, \quad y \cdot n > a\}$ and ϕ is 1-periodic in y.

We expect that the solution U of (BL) satisfies:

$$U \to U_\infty(\phi), \qquad \text{as } y \cdot n \to +\infty,$$

for some constant $U_\infty = U_\infty(\phi)$ that depends linearly on ϕ.

If we go back to U_0 which solves (7.2), one can derive the homogenized boundary data φ^0. Indeed:

- On one hand, one wants $U_0 \to 0$ (localization property) when $y \cdot n \to +\infty$.
- On the other hand,

  $$U_0 \to U_\infty(\varphi - \varphi^0(x_0)) \;=\; U_\infty(\varphi) - \varphi^0(x_0)$$

  so that we need to take:

  $$\varphi^0(x_0) \;:=\; U_\infty(\varphi).$$

This formal reasoning raises many problems :

1. *The well-posedness of* (BL) *is unclear:*

   - No natural functional setting (no decay along the boundary).
   - No Poincaré inequality.
   - No maximum principle.

2. *The existence of a limit U_∞ for* (BL) *is unclear:*

   There is an underlying problem of ergodicity.

3. *U_∞ depends also on H, that is on n and a:*

   - There is no obvious regularity of U_∞ with respect to n.

   - Back to the original problem, our definition of $\varphi^0(x_0)$ depends on x_0, but also on the subsequence ε. Indeed, there is possibly many accumulation points as $\varepsilon \to 0$ (see [19]).

7.2 Polygons with sides of rational slopes

In this cases, the boundary layer systems of type (BL) can be fully understood (see [19, 2]). For simplicity, we only concentrate on the case $d = 2$.

1. Well-posedness: *The coefficients of the systems are periodic tangentially to the boundary.* After rotation, they turn into systems of the type

$$\boxed{\begin{cases} \nabla_z \cdot (B\nabla_z V) = 0, \quad z_2 > a, \\ V|_{z_2=a} = \psi, \end{cases}} \tag{BL1}$$

   with coefficients and boundary data that are periodic in z_1 which yields a natural variational formulation.

2. Existence of the limit : *Saint-Venant estimates* on (BL1).

   One shows that $F(t) := \int_{z_2>t} |\nabla_z V|^2 \, dz$ satisfies the differential inequality.

$$F(t) \le -CF'(t).$$

   ¿From there, one gets exponential decay of all derivatives, and the fact that:

$$V \to V_\infty, \text{ exponentially fast, as } z_2 \to +\infty$$

   and hence going back to (BL), we get

$$U \to U_\infty, \text{ exponentially fast, as } y \cdot n \to +\infty.$$

   A Key ingredient in this case is the Poincaré inequality for functions periodic in z_1 with zero mean.

3. In polygonal domains, the regularity of U_∞ with respect to n does not matter. However, for rational slopes, the limit U^∞ does depend on a. This means that if we go back to our original problem (in polygons with rational slopes), The analogue of our theorem is only available up to subsequences in ε. Moreover, the boundary data of the homogenized system may depend on the subsequence. Indeed, there are examples with a continuum of accumulation points (see [19]).

7.3 More general treatment of (BL)

It is worth pointing out that one can not be fully general: The existence of U_∞ requires some ergodicity property. A simple example is :

$$\boxed{\Delta U = 0 \quad \text{in } \{y_2 > 0\}, \quad U|_{y_2=0} = \phi}.$$

- If ϕ 1-periodic, then $U(0, y_2) \to \int_0^1 \phi$ exponentially fast.
- *But there exists $\phi \in L^\infty$ such that $U(0, y_2)$ has no limit.*

Indeed, we have an explicit formula: $U(0, y_2) = \dfrac{1}{\pi} \displaystyle\int_{\mathbb{R}} \frac{y_2}{y_2^2 + t^2} \phi(t)\, dt$. For ϕ with values in $\{+1, -1\}$, the asymptotics relates to *coin tossing*. Hence, we need some extra structure (or ergodicity) to solve the problem.

In our case, we have some ergodicity property ! For general half planes, the coefficients of (BL) or (BL1) are not periodic, *but they are quasiperiodic in the tangential variable.* We recall that a function $F = F(z_1)$ is quasiperiodic if it reads

$$\boxed{F(z_1) = \mathcal{F}(\lambda z_1)},$$

where $\lambda \in \mathbb{R}^D$ and $\mathcal{F} = \mathcal{F}(\theta)$ is periodic over $\mathbb{R}^D$ ($D \geq 1$). As an example: For (BL1), $D = 2$, and $\lambda = n^\perp$ (the tangent vector).

Notice that the previous results (subsection 7.2) correspond to the case: $n \in \mathbb{R}\,\mathbb{Q}^2$. Now, we replace this by the small divisor assumption:

$$\boxed{\text{(H)} \quad \exists \kappa > 0,\ |n \cdot \xi| \geq \kappa |\xi|^{-2}, \quad \forall \xi \in \mathbb{Z}^2 \setminus \{0\}.}$$

Note that the assumption (H) is generic in the normal n: It is satisfied by a set of full measure in $\mathbb{S}^1$. But it does not include the previous result of subsection 7.2.

Proposition 2 If n satisfies (H), the system (BL) is "well-posed", with a smooth solution U that converges fast to some constant U_∞. Moreover, U_∞ does not depend on a.

Proof of the proposition:

1. Well-posedness: involves quasiperiodicity. One has:

$$\boxed{\begin{cases} \nabla_z \cdot (B \nabla_z V) = 0, & z_2 > a, \\ V|_{z_2 = a} = \psi, \end{cases}}$$

   where $B(z) = \mathcal{B}(\lambda z_1, z_2)$, $\psi(z) = \mathcal{P}(\lambda z_1, z_2)$.

   Notice that the functions $\mathcal{B} = \mathcal{B}(\theta, t)$ and $\mathcal{P} = \mathcal{P}(\theta, t)$ are periodic in $\theta \in \mathbb{T}^2$.

   The idea is to consider an enlarged system in θ, t, of unknown $\mathcal{V} = \mathcal{V}(\theta, t)$:

$$\boxed{\begin{cases} D \cdot (\mathcal{B} D \mathcal{V}) = 0, & t > a, \\ \mathcal{V}|_{t=a} = \mathcal{P} \end{cases}} \tag{BL2}$$

where D is the "degenerate gradient" given by $D = (\lambda \cdot \nabla_\theta, \partial_t)$

Advantage: Back to a periodic setting ($\theta \in \mathbb{T}^2$).

Drawback: We have a degenerate elliptic equation. However, we are still able to prove the following :

- Variational formulation with a unique weak solution $\mathcal{V}$.

- One can prove through energy estimates than $\mathcal{V}$ is smooth.

- Allows to recover V through the formula $V(z) = \mathcal{V}(\lambda z_1, z_2)$.

2. To prove the convergence to a constant at infinity, we rely again on Saint-Venant type estimates, adapted to (BL2). Thanks to (H), we prove that $F(t) := \int_{t'>t} |D\mathcal{V}|^2 \, d\theta \, dt'$ satisfies

$$\boxed{F(t) \le C(-F'(t))^\alpha, \quad \forall \alpha < 1.}$$

But, we have only polynomial convergence towards a constant.

We point out that this better understanding of the auxiliary boundary layer systems allows to handle the generic polygonal domains in the next subsection.

7.4 Extension to smooth domains

The are at least three main difficulties to extend the previous analysis to smooth domain :

1. The none smoothness of U_∞ with respect to n. Indeed, U_∞ is only defined almost everywhere (diophantine assumption).

   Idea: *For any $\kappa > 0$, we can prove that U_∞ is Lipschitz when it is restricted to*

$$A_\kappa := \left\{ n \in \mathbb{S}^1, \, |n \cdot \xi| \ge \frac{\kappa}{|\xi|^2}, \, \forall \xi \in \mathbb{Z}^2 \setminus \{0\} \right\}.$$

   Moreover, we have that $|A_\kappa^c| = O(\kappa)$.

   In the course of the proof, the construction of the boundary layer corrector can be performed in the vicinity of points x such that $n(x) \in A_\kappa$. In some sense, the contribution of the remaining part of the boundary is negligible when $\kappa \ll 1$. More precisely,

2. We have to approximate the smooth domains by some polygons with sides having normal vectors in the set A_κ. In doing so, we will introduce another small parameter ε^α.

3. We have to construct a more accurate approximation due to the many errors made in the previous two points.

Broadly, optimizing in κ, α and ε yields a rate of convergence. We refer to [9] for the details.

8 Conclusions

We would like to conclude by mentioning a few related results. Recently there was many activity in the theory of homogenization and many new problems were addressed. We would like to mention some of them since we think they may give a better understand of our result or/and may be combined with our result:

- Our results on the boundary data problem were recently extended to the eigenvalue problem, see [21]. Also, the behavior of the reduced boundary layer system (BL) was recently investigated by C. Prange in [22], without any diophantine assumption.
- The Avellaneda-Lin type estimates were extended to the case of Neumann boundary conditions by Kenig, Lin and Shen [14, 15, 16] (see also [5] for a related work). These estimates should be helpful to study the next order approximation for the Neumann boundary condition case
- Many new probabilistic results were proved when an interface is present (see [12]) or in the trying to compute the accurate value of the homogenized matrix (see [10, 11]).
- Some different method was used to compute homogenized boundary data for none oscillating coefficient ([17, 1]).

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